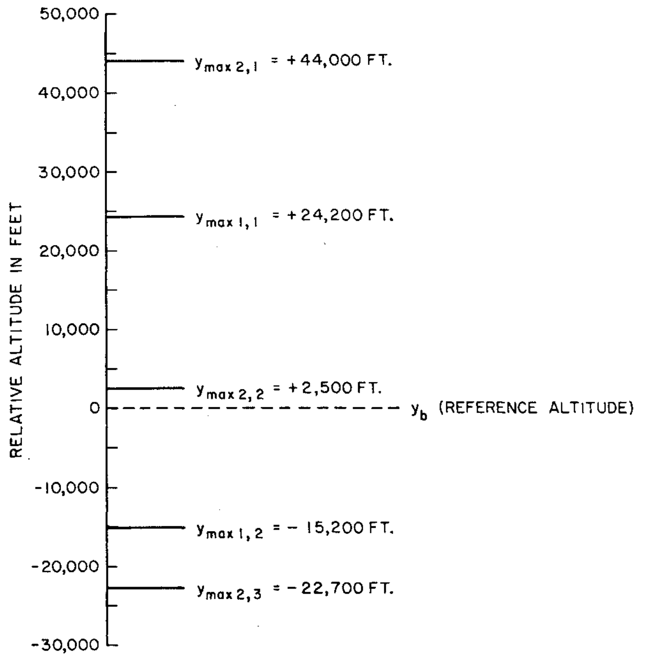

Fig. 3 Relative altitudes at which the peak values of time derivatives occur ($\beta = 1/22{,}000$ ft^{-1})

gives a cubic equation, which, when solved and combined with Eq. (3), gives

$$y_{\max 2,1} = \frac{1}{\beta} \log_e \frac{7.52\, C_D \rho_0 A}{\beta m \sin\theta_E} = y_{\max} + \frac{2.02}{\beta} \tag{11}$$

$$y_{\max 2,2} = \frac{1}{\beta} \log_e \frac{1.12\, C_D \rho_0 A}{\beta m \sin\theta_E} = y_{\max} + \frac{0.113}{\beta} \tag{12}$$

$$y_{\max 2,3} = \frac{1}{\beta} \log_e \frac{C_D \rho_0 A}{2.81 \beta m \sin\theta_E} = y_{\max} - \frac{1.03}{\beta} \tag{13}$$

Again it is seen that, for a constant β, the vertical distances between the altitude of maximum deceleration and the altitudes of maximum and minimum values for the second derivative of deceleration are constants. To determine the maximum and minimum values of d^3V/dt^3, substitute the values for $y_{\max}$ in Eqs. (11–13) back in Eq. (6). This yields

$$d^3V/dt^3{}_{y_{\max 2,1}} = -0.0286\, \beta^3 V_E{}^4 \sin^3\theta_E \tag{14}$$

$$d^3V/dt^3{}_{y_{\max 2,2}} = 0.070 \beta^3 V_E{}^4 \sin^3\theta_E \tag{15}$$

$$d^3V/dt^3{}_{y_{\max 2,3}} = -0.0153 \beta^3 V_E{}^4 \sin^3\theta_E \tag{16}$$

Again, it is seen that these values are independent of the drag characteristics of the object.

Graphic Results

Figure 1 is a plot of the deceleration and its two higher time derivatives as a function of altitude for a typical object entering the atmosphere with the following parameters: V_E = 25,000 fps, θ_E = 40°, W/C_DA = 100 psf, and β = 1/22,000 ft^{-1}. Deceleration has been taken as equal to $-dV/dt$ and thus appears as a positive quantity. In a similar fashion, the time rate of change of deceleration equals $-d^2V/dt^2$, and the second time derivative of deceleration equals $-d^3V/dt^3$.

The other curves facilitate determination of the peak magnitudes of deceleration and its time derivatives and the altitudes at which these peaks occur. Figure 2 gives the altitudes of peak deceleration as a function of entry angle and object ballistic coefficient, W/C_DA. As shown in Eq. (3), this altitude is independent of entry velocity.

Figure 3 gives the relative altitudes of the peaks of the higher time derivatives referenced to the altitude of peak deceleration assuming that β = 1/22,000 ft^{-1}. From Figs.

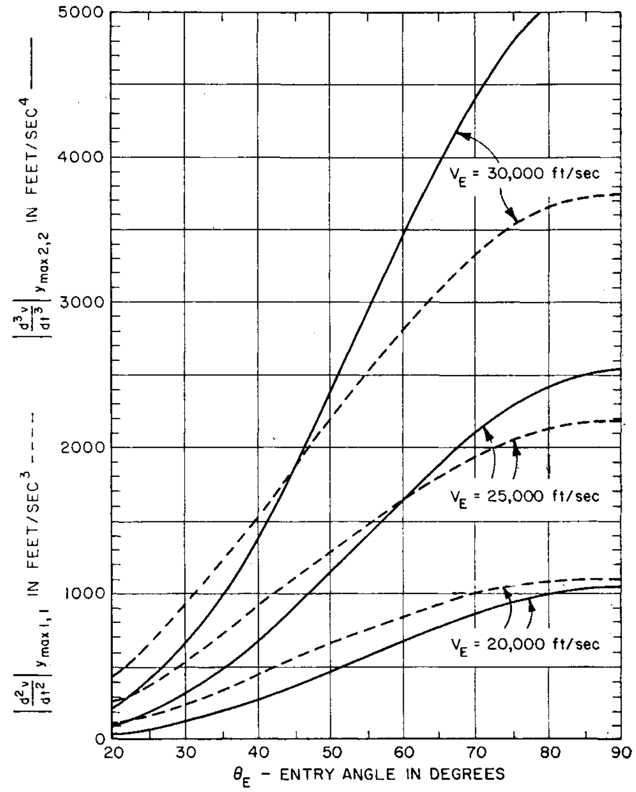

Fig. 4 Maximum magnitudes of time derivatives

2 and 3 the altitude at which any maximum or minimum of the two higher time derivatives of deceleration occurs may be determined.

Figure 4 gives the maximum magnitudes reached for the two higher time derivatives of deceleration as a function of entry angle and velocity. These values are independent of the drag characteristic of the object.

Thermal Deflection of a Circular Sandwich Plate

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A differential equation is derived for the thermal deflection of a circular sandwich plate. The effect of core shear rigidity is included in the analysis. A particular solution is obtained for a simply supported plate with uniform and unequal temperatures of the two face plates. In addition, expressions are presented for the radial and circumferential bending stresses. It is shown that the plate center deflection is dependent on core rigidity only for values of the rigidity parameter $\bar{X}$ less than 10. Beyond this value the deflection is essentially the same as with infinite core shear rigidity.

SOME assumptions of this analysis are 1) axial symmetry ($\partial/\partial_\theta = 0$), 2) $\sigma_r = 0$ and $\sigma_\theta = 0$ in the core, 3) core is assumed isotropic with shear rigidity G_c psi, and 4) transverse

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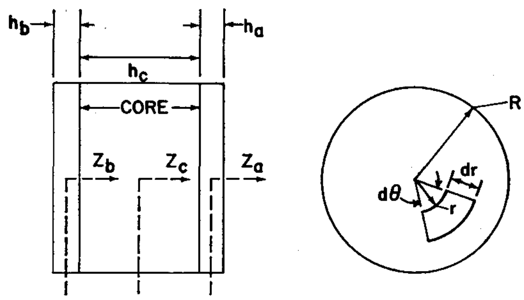

Fig. 1 Composite plate geometry

normal deformation is neglected ($\partial w/\partial z = 0$). The symbolic notation is the same as used in a previous paper.[1] Figure 1 illustrates the plate under consideration. Figure 2 shows loading conditions of the two face elements and of a core element, respectively.

From equilibrium of the core element (see Fig. 2), one obtains

$$Q_c = \tau h_c \tag{1}$$

$$P_a + \tau h_c/r + d\tau/dr\, h_c = P_b \tag{2}$$

For the radial displacement of a core element one obtains

$$u_c = \tau Z_c/G_c - Z_c\, dw/dr \tag{3}$$

(The absence of core normal stresses σ_r and σ_θ, together with the condition $\partial w/\partial z = 0$, makes the solution for core displacement indifferent to core temperature distribution.)

Assuming no slip in the face-core bond, one has

$$u_a(r, -h_a/2) - u_b(r, +h_b/2) = u_c(r,h_c/2) - u_c(r, -h_c/2) \tag{4}$$

Substituting Eqs. (3) and (14)*† into Eq. (4), one obtains

$$u_{0a} = u_{0b} - (h_c + h_a/2 + h_b/2)(dw/dr) + \tau h_c/G_c \tag{5}$$

Let $h_c + h_a/2 + h_b/2 = h_0$,

$$Q_a = dM_{ra}/dr + (M_{ra} - M_{\theta a})/r + h_a\tau/2 \qquad (8)^*‡$$

$$Q_b = dM_{rb}/dr + (M_{rb} - M_{\theta b})/r + h_b\tau/2 \qquad (11)^*$$

Modifying Eqs. (9)* and (12)* for this analysis, one has

$$dQ_a/dr + Q_a/r - P_a = 0 \qquad (9)^*$$

$$dQ_b/dr + Q_b/r + P_b = 0 \qquad (12)^*$$

By substituting (8)* and (11)* together with Eqs. (18*, 19*, and 2) into (9)* and (12)* and adding, one obtains

$$-(D_a + D_b)\nabla^4 w + h_0(\tau/r + d\tau/dr) = (1/1 - v)(\nabla^2 Ma^* + \nabla^2 M_b^*) \tag{6}$$

where

$$\nabla^2 = 1/r\, d/dr\, (r\, d/dr)$$
$$\nabla^4 = 1/r\, d/dr\, (r\, d/dr\, \nabla^2)$$

Substituting Eqs. (16*, 17*, and 5) into (7)* and (10)*, upon adding and subtracting, one obtains

$$(K_a + K_b)[d/dr(du_{0a}/dr + u_{0a}/r)] + k_b h_0\, d/dr\, \nabla^2 w - K_b\, h_c/G_c\, [d/dr\, (d\tau/dr + \tau/r)] - (1/1 - v)\, d/dr\, (T_a^* + T_b^*) = 0 \tag{7}$$

[1] Vinson, J. R., "Thermal stresses in laminated circular plates," *Proceedings of the Third U. S. Congress of Applied Mechanics* (American Society of Mechanical Engineers, New York, 1958), p. 467.

† Equations marked with asterisks refer to same equation numbers in footnote 1.

‡ This equation shows a plus sign for the (τ) term, differing from Eq. (8) in footnote 1 which appears to be misprinted.

and

$$(K_a - K_b)[d/dr\, (du_{0a}/dr + u_{0a}/r)] - K_b h_0\, d/dr\, \nabla^2 w - 2\tau + K_b h_c/G_c[d/dr(d\tau/dr + \tau/r)] - (1/1 - v)\, d/dr\, (T_a^* - T_b^*) = 0 \tag{8}$$

Eliminating $du_{0a}/dr + u_{0a}/r$ by combination of Eqs. (7) and (8), one obtains

$$[-2K_aK_b/(K_a + K_b)]h_0\, d/dr\, \nabla^2 w - 2\tau + [2K_aK_bh_c/(K_a + K_b)\, G_c][d/dr\, (d\tau/dr + \tau/r)] - (1/1 - v)[2/(K_a + K_b)](K_b\, dT_a^*/dr - K_a\, dT_b^*/dr) = 0 \tag{9}$$

Rewriting Eq. (6),

$$h_0(1/r)\, d/dr\, (r\tau) = (D_a + D_b)(1/r)\, d/dr\, (r\, d/dr\, \nabla^2 w) + (1/1 - v)[1/r\, d/dr\, (r\, d/dr)(M_a^* + M_b^*)]$$

By integrating,

$$\tau = (D_a + D_b)/h_0\, d/dr\, \nabla^2 w + (1/1 - v) \times (1/h_0)[d/dr\, (M_a^* + M_b^*)] + C_1/r \tag{10}$$

Substituting (10) and (6) into (9),

$$[-2K_aK_b/(K_a + K_b)]h_0\, d/dr\, \nabla^2 w - 2(D_a + D_b)/h_0\, d/dr\, \nabla^2 w - 2C_1/r + [2K_aK_bh_c/(K_a + K_b)G_c][(D_a + D_b)/h_0](d/dr)(\nabla^4 w) + 2K_aK_bh_c/(K_a + K_b)G_c\, [1/(1 - v)h_0](d/dr)[\nabla^2(M_a^* + M_b^*)] - (2/1 - v)(1/K_a + K_b)(d/dr)(K_bT_a^* - K_aT_b^*) - (2/1 - v)(1/h_0)[d/dr\, (M_a^* + M_b^*)] = 0 \tag{11}$$

By collecting terms,

$$-[(2K_aK_b/K_a + K_b)h_0 + 2(D_a + D_b)/h_0]\, d/dr\, \nabla^2 w - 2C_1/r + 2K_aK_bh_c/(K_a + K_b)G_c[D_a + D_b/h_0]\, d/dr\, \nabla^4 w + [2K_aK_bh_c/(K_a + K_b)G_c][1/(1 - v)h_0]\, d/dr\, \nabla^2(M_a^* + M_b^*) - (2/1 - v)(1/K_a + K_b)\, d/dr\, (K_bT_a^* - K_aT_b^*) - (2/1 - v)(1/h_0)[d/dr\, (M_a^* + M_b^*)] = 0$$

Let

$$B_1 = 2K_aK_bh_c(D_a + D_b)/(K_a + K_b)G_ch_0$$
$$B_2 = (2K_aK_bh_0/K_a + K_b) + 2(D_a + D_b)/h_0$$

Equation (11) becomes

$$-B_2\, d/dr\, \nabla^2 w + B_1\, d/dr\, \nabla^4 w + [B_1/(D_a + D_b) \times (1 - v)]\, d/dr\, \nabla^2(M_a^* + M_b^*) - (2/1 - v) \times (1/K_a + K_b)\, d/dr\, (K_bT_a^* - K_aT_b^*) - 2C_1/r - (2/1 - v)(1/h_0)[d/dr\, (Ma^* + M_b^*)] = 0$$

Upon integrating three times, one obtains

$$-B_2\, w + B_1\, d^2w/dr^2 + B_1/r\, dw/dr = f(r) = (C_1r/2)(\ln r) + r^2/4(C_2 - 2C_1) + C_3\ln r + C_4 + [2/(1 - v)h_0]\int 1/r[\int r\, (M_a^* + M_b^*)\, dr]\, dr + [2/(1 - v)(K_a + K_b)]\int 1/r \int [r(K_bT_a^* - K_aT_b^*)] \times dr\, dr - [B_1/(D_a + D_b)(1 - v)][(M_a^* + M_b^*)] \times$$

$$d^2w/dr^2 + 1/r\, dw/dr - B_2/B_1\, w = f(r)/B_1$$

Let

$$x = ar \qquad a^2 = B_2/B_1$$
$$d^2w/dx^2 + 1/x\, dw/dx - w = f(r)/B_1$$

The solution of the homogeneous equation is

$$w_1 = C_5 I_0(x) \qquad w_2 = C_6 K_0\, (x)$$

Where I_0 and K_0 are the modified Bessel functions of the first and second kind, respectively.

Obtain a particular solution for simply supported plate

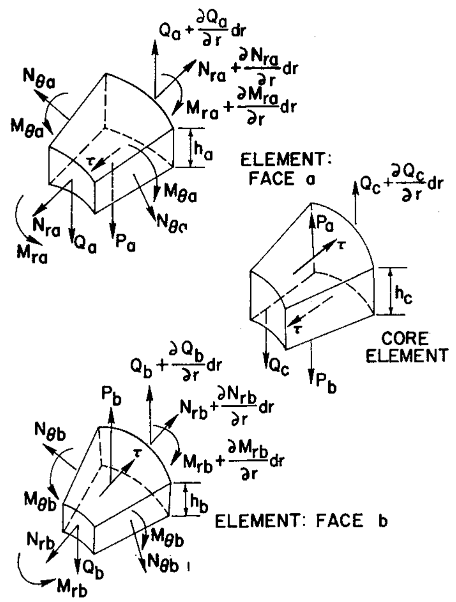

Fig. 2 Internal stress resultants of face and core elements

with T_a and T_b independent of r and z ($M_a^* = 0$, $M_b^* = 0$); thus,

$$f(r) = (C_1r^2/2) \ln r + r^2/4(C_2 - 2C_1) + C_3 \ln r + C_4 + [2/(1 - v)(K_a + K_b)] \times (K_bT_a^* - K_aT_b^*)(r^2/4)$$

Let

$$\tfrac{1}{4}\{C_2 - 2C_1 + [2/(1 - v)(K_a + K_b)][(K_bT_a^* - K_aT_b^*)]\} = S$$

Therefore

$$f(r) = (C_1r^2/2)(\ln r) + Sr^2 + C_3 \ln r + C_4$$

Boundary Conditions

For $r = 0$,

$$dw/dr = 0 \qquad u = 0$$

Therefore $C_1 = 0$,

$$1/r\, dw/dr - d^2w/dr^2 - r\, d^3w/dr^3 = 0 \qquad (Q_a + Q_b = 0)$$

For $r = R$,

$$w = 0$$

$$vdw/dr + R\, d^2w/dr^2 = 0 \qquad (M_r = 0,\ M^* = 0)$$

$$vu/R + du/dr = T^*/(1 - v)K \qquad (N_{ra} = 0,\ N_{rb} = 0;\ N_{ra} - N_{rb} = 0)$$

Therefore

$$(K_bK_a)(h_c/G_c)(D_a + D_b/h_0)d^2/dr^2(d^2w/dr^2 + 1/r\, dw/dr) = -(1/1 - v)(K_bT_a^* - K_aT_b^*)$$

Employing the method of variation of parameters and substituting the boundary conditions, it can be shown that the expression for the plate center deflection is

$$W_{r=0} = \{\alpha(T_a - T_b)(1 + v)(R^2)/[1 + \tfrac{1}{3}(h/h_0)^2](h_0)\} \times \{(\bar{I}_0 - 1)/\bar{X}^2 - [\tfrac{1}{2}(1 + v)][\bar{I}_0 - \bar{I}_1(1 - v)/\bar{X}]\} \times [1/(\bar{I}_0 - \bar{I}_1/\bar{X})]$$

where

$$\bar{X} = aR \qquad \bar{I}_0 = I_0(\bar{X}) \qquad \bar{I}_1 = I_1(\bar{X})$$

and

$$h_a = h_b = h \qquad K_a = K_b \qquad D_a = D_b \qquad T_a - T_b = \Delta T$$

$$a^2 = [1 + \tfrac{1}{3}(h/h_0)^2][6G_c(h_0/h)^2(1 - v^2)/E\, h\, (h_0 - h)]$$

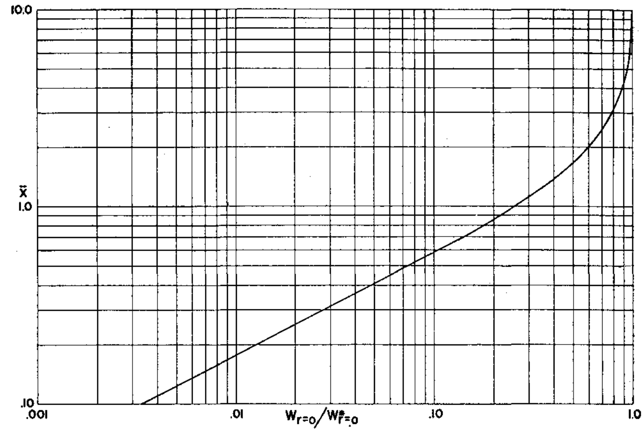

Fig. 3 Effect of core rigidity on deflection (nondimensionalized)

It can be shown that if $a \to \infty$ the expression for $W_{r=0}$ becomes

$$W_{r=0} \text{ with } a \to \infty = W_{r=0}^* = \alpha(\Delta T)\, R^2/2[1 + (\tfrac{1}{3})(h/h_0)^2](h_0)$$

Figure 3 is a plot of $W_{r=0}/W_{r=0}^*$ vs $\bar{X}$ for $v = 0.3$.

Bending Stresses: Particular Solution

$$M_r = -D(d^2w/dr^2 + v/r\, dw/dr) - 1/1 - v\, M^* \qquad (18^*)$$

$$M^* = 0 \text{ for particular case}$$

$$\sigma_r = 6M_r/h^2$$

It may be shown that

$$\sigma_r = \{E\alpha(\Delta T)/2(1 - v)(h_0/h)[1 + \tfrac{1}{3}(h/h_0)^2]\} \times \{[I_0 - \bar{I}_0 + (1 - v)(\bar{I}_1/\bar{X} - I_1/X)]/(\bar{I}_0 - \bar{I}_1/\bar{X})\}$$

Similarly,

$$M_\theta = -D(1/r\, dw/dr + v\, d^2w/dr^2) - 1/1 - v\, M^*$$

$$\sigma_\theta = 6M_\theta/h^2 \qquad (19^*)$$

$$\sigma_\theta = \{E\alpha(\Delta T)/2(1 - v)(h_0/h)[1 + \tfrac{1}{3}(h/h_0)^2]\} \times \{[vI_0 - \bar{I}_0 + (1 - v)(I_1/X + \bar{I}_1/\bar{X})]/(\bar{I}_0 - \bar{I}_1/\bar{X})\}$$

where

$$I_0 = I_0(X) \qquad \bar{I}_0 = I_0(\bar{X})$$
$$I_1 = I_1(X) \qquad \bar{I}_1 = I_1(\bar{X})$$
$$x = ar \qquad \bar{X} = aR$$

A Closed Form Solution of the Relativistic Differential Equation for Planetary Motion

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1. Introduction

AN outline of the derivation of Einstein's general relativistic differential equations for the motion of a particle in a gravitational field has been given by Lindsay and Margenau[3] and Sokolnikoff.[6] A detailed exposition of the derivation of these equations has been given by the author.[1]

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